# ON A PROBLEM OF THE THEORY OF DYNAMIC PROGRAMMING

## (OB ODNOI ZADACHE TEORII DINAMICHESKOGO PROGRAMMIROVANIIA)

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E. A. BARBASHIN
(Sverdlovsk)

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Methods for the selection of the input parameters of linear systems are given. The object of these methods is to insure the transition of the system from the given initial state to a new nearby state.

**1.** Let us consider the $n$th-order linear differential equation

$$L(x) = x^{(n)} + a_1(t)\,x^{(n-1)} + \ldots + a_n(t)\,x = c_1u_1(t) + \ldots + c_mu_m(t) \qquad (1.1)$$

where $a_1(t), \ldots, a_n(t)$ are continuous functions of time for $t \geqslant 0$; $u_1(t), \ldots, u_m(t)$ are a given set of linearly independent functions; $c_1, \ldots, c_m$ are constant parameters which can be chosen within certain limits.

Suppose that at $t = 0$ we are given the set of numbers $x_0, x_0', \ldots, x_0^{(n-1)}$ and suppose that $f(t)$ is a given function defined on $0 \leqslant t \leqslant T$, $0 < T \leqslant \infty$. We state two problems:

1) The problem is to find a set of parameters $c_i$ such that the solution $x(t)$ of Equation (1.1) satisfying the conditions

$$x(0) = x_0, \qquad x'(0) = x_0', \ldots, x^{(n-1)}(0) = x_0^{(n-1)} \qquad (1.2)$$

may also satisfy the condition

$$x(t_0) = f(t_0), \qquad x'(t_0) = f'(t_0), \ldots, x^{(n-1)}(t_0) = f^{(n-1)}(t_0) \qquad (1.3)$$

when $t = t_0 > 0$.

2) The second problem is to find a set of parameters $c_i$ such that the solution $x(t)$ of Equation (1.1) which satisfies (1.3) may approximate the given function $f(t)$ on the set $t_0 \leqslant t \leqslant T$.

If one solves these problems simultaneously, one is looking for piecewise constant functions $c_i(t)$ which change their values when $t = t_0$ and which guarantee the transition of the system at the time $t_0$ into a new state with an ultimate approximate realization of the given process $f(t)$. During the time $0 \leqslant t \leqslant t_0$ a transient process is taking place which begins at $t_0$ with a new set of parameters $c_i$. This new set of parameters must approximate as much as possible the solution $x(t)$ of Equation (1.1) to the given function $f(t)$.

In solving the first problem we shall endeavor to find the smallest $t_0$ for which the given boundary-value problem has a solution. While solving the first and second problems we shall remember also that in applied problems one cannot select the parameters $c_i$ arbitrarily, for they are restricted by the structural characteristics of the system under consideration. These circumstances put the first problem into the class of problems on optimum control with respect to speed. Krasovskii [1] was the first to call attention to the possibility of applying Krein's $L$-problem theory to the given class of problems. This approach is used in the present paper. We note that the search for the optimal control in the form of a trigonometric polynomial was carried out by Krasovskii in [2].

The second problem is a problem in the theory of approximations. It has been considered, in particular, by Kulinovskii [3,4]. In the present article a different method of solution is used from that given in the indicated works. Following the ideas expressed in [5,6], one can avoid computational difficulties by replacing the problem on best approximation of the function $f(t)$ by the problem of finding such parameters $c_i$ for which the function $f(t)$ satisfies Equation (1.1) with the least error.

Let $w_1(t, \tau), \ldots, w_n(t, \tau)$ be a linearly independent system of solutions of Equation (1.1) satisfying the conditions

$$\left.\frac{d^k w_i(t, \tau)}{dt^k}\right|_{t=\tau} = \delta_{i,\,k+1} \qquad (\delta_{i,\,k+1} \text{ is Kronecker's symbol}) \tag{1.4}$$

The solution of Equation (1.1) which satisfies Equation (1.2) can be expressed in the form [7]

$$x(t) = \sum_{k=1}^{n} w_k(t, 0)\, x_0^{(k-1)} + \sum_{i=1}^{m} c_i \int_0^t w_n(t, \tau)\, u_i(\tau)\, d\tau \tag{1.5}$$

Let

$$y_i(t) = \int_0^t w_n(t, \tau)\, u_i(\tau)\, d\tau \qquad (i = 1, \ldots, m) \tag{1.6}$$

It is not difficult to select the functions $u_i(\tau)$ in such a way that the $m+n$-functions $w_1(t, 0), \ldots, w_n(t, 0), y_1(t), \ldots, y_m(t)$ be linearly independent.

For the purpose of justifying our formulation of the problem, we digress from our main aim, and first attempt to select the parameters $c_i$ and the initial values $x_0, x_0', \ldots, x_0^{(n-1)}$ in such a way that the solution of Equation (1.1)

$$x(t) = \sum_{k=1}^{n} w_k(t, 0)\, x_0^{(k-1)} + \sum_{i=1}^{m} c_i y_i(t) = \sum_{i=1}^{m+n} b_i z_i(t) \qquad (1.7)$$

where

$$a_i = x_0^{(i-1)}, \quad z_i(t) = w_i(t, 0) \quad \text{when } 1 \leqslant i \leqslant n$$

$$a_i = c_{i-n}, \quad z_i(t) = y_{i-n}(t) \quad \text{when } n < i \leqslant m$$

will be the best approximation to the given function $f(t)$ on the interval $[0, T]$.

When one speaks of the best approximation in the space $L_2$, that is, when one requires that the quantity

$$H^2 = \int_0^T (x(t) - f(t))^2\, dt$$

be a minimum, then it follows from the theory of mean-square approximations [4,8] that the initial values and parameters can be found by means of the system

$$\sum_{k=1}^{m+n} (z_i, z_k)\, b_k = (z_i, f) \qquad (i = 1, \ldots, m+n) \qquad (1.8)$$

Here

$$(z_i, z_k) = \int_0^T z_i(t)\, z_k(t)\, dt, \qquad (z_i, f) = \int_0^T z_i(t)\, f(t)\, dt$$

For the indicated choice of the $b_k$ we will have

$$H^2 = \frac{\Gamma(z_1, \ldots, z_{n+m}, f)}{\Gamma(z_1, \ldots, z_{n+m})}$$

where the numerator and denominator are the Grammian determinants of the corresponding systems of functions.

The problem becomes considerably more complicated when one looks for the best approximation in the space $C$, i.e. if one requires a minimum for the quantity

$$h = \max |x(t) - f(t)| \qquad (0 \leqslant t \leqslant T)$$

The theory of uniform or of Chebyshev approximations does not yield any methods which are as simple as the above-described procedure for finding the $b_k$.

Let us denote by $x_1(t)$ a solution of Equation (1.1) which satisfies the conditions

$$x_0^{(k)}(t_0) = f^{(k)}(t_0) \qquad (k = 0, 1, \ldots, n-1)$$

We have, obviously

$$|x(t) - f(t)| \leqslant |x(t) - x_1(t)| + |x_1(t) - f(t)|$$

Since

$$x(t) - x_1(t) = \sum_{k=1}^{n} w_k(t, t_0)\,(x^{(k-1)}(t_0) - f^{(k-1)}(t_0))$$

the solution of the problem can be carried out in two stages. At the first stage we shall attempt to eliminate the difference between the actual and the desired initial conditions of the system (i.e. we accomplish the transient process). At the second stage, we select new values of the parameters; we attempt to diminish the difference between the actual $x(t)$ and the desired $f(t)$ processes. The indicated stages correspond exactly to the above-stated first and second problems.

We call attention to the fact that the control $c_1u_1(t) + \ldots + c_mu_m(t)$ for which we are searching need not be expressible as an explicit function in $t$. Indeed, the function $c_1u_1 + \ldots + c_mu_m(t)$ obviously satisfies some linear equation $L_1(u) = 0$ of order $m$. The problem on the determination of the parameters $c_i$ can be formulated in this case as the problem on the finding of the initial values for the solution of the indicated equation.

**2.** Let us proceed with the solution of the first problem without putting, for the time being, any restrictions on $c_i$. Making the change of variables $z = x - f(t)$ in Equation (1.1), we obtain

$$L(z) = z^{(n)} + a_1(t)z^{(n-1)} + \ldots + a_n(t)z = c_1u_1(t) + \ldots + c_mu_m(t) - L(f(t)) \tag{2.1}$$

Taking into consideration conditions (1.2) and Formulas (1.5) and (1.6), one can write the solution of Equation (2.1) in the form

$$z(t) = \sum_{i=1}^{m} c_i y_i(t) - r(t) \qquad \left(r(t) = f(t) - \sum_{k=1}^{n} w_k(t, 0)\, x_0^{(k-1)}\right) \qquad (2.2)$$

When $t = 0$, this equation satisfies the condition

$${}^{(k)}(0) = x_0^{(k)} - f^{(k)}(0) \qquad (k = 0, 1, \ldots, n-1) \qquad (2.3)$$

Differentiating (2.2) $n - 1$ times, we obtain

$$z^{(k)}(t) = \sum_{i=1}^{m} c_i y_i^{(k)}(t) - r^{(k)}(t) \qquad (k = 0, 1, \ldots, (n-1)) \qquad (2.4)$$

The problem consists of selecting such values of the parameters $c_i$ that for $t = t_0 > 0$ the equation

$$\sum_{i=1}^{m} c_i y_i^{(k)}(t_0) = r^{(k)}(t_0) \qquad (k = 0, 1, \ldots, (n-1)) \qquad (2.5)$$

may have a solution.

Since the system (2.5) is inconsistent, we must find a solution by the method of least squares [6, p. 449], i.e. we must look for such a set of parameters $c_i$ that the quadratic form

$$F = \sum_{k=0}^{n-1} \left(\sum_{i=1}^{m} c_i y_i^{(k)} - r^{(k)}\right)^2 \qquad (2.6)$$

in these parameters may have a minimum (here and in the sequel we omit the argument $t_0$).

Next, let us consider the vectors

$$Y_i(y_i, y_i', \ldots, y_i^{(n-1)}) \quad (i = 1, \ldots, m).$$

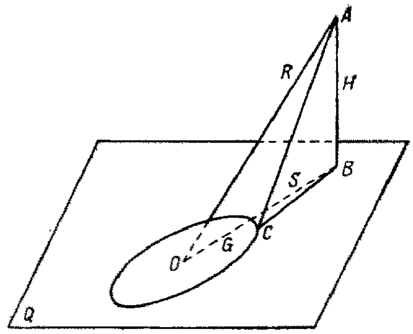


Fig. 1.

Since they are $n$-dimensional vectors, there exist among them $p$ linearly independent vectors (here $p \leqslant n$, $p \leqslant m$). We shall denote these vectors by $Y_1, \ldots, Y_p$. By $Q$ we shall denote the hyperplane generated by the given vectors. Obviously, all remaining vectors $Y_i$ lie in this hyperplane and the quadratic form $F$ is the square of the distance from the point $A$, whose radius vector is equal to $R(r, r', \ldots, r^{(n-1)}$, to the point $B$ with radius vector $S = c_1 Y_1 + \ldots + c_m Y_m$ lying in the plane $Q$ (see Fig. 1). But then $F$ must be a minimum if the vector $S$ is the projection of the vector $R$, or, which is the same thing, the point $B$ will be the projection of the point $A$. Since the system of vectors $Y_1, \ldots, Y_p$ is a basis for the subspace $Q$, we have

$$S = c_1 Y_1 + \ldots + c_p Y_p$$

In accordance with [10, p. 204], the parameters $c_i (1 \leqslant i \leqslant p)$ are here determined by the system

$$(Y_i, Y_1) c_1 + \ldots + (Y_i Y_p) c_p = (Y_i, R) \qquad (i = 1, \ldots, p) \tag{2.7}$$

Here, $(Y_i, Y_k)$ stands for the inner product of the vectors $Y_i$ and $Y_k$. Thus, the $c_i$ with $i > p$ do not enter into the solution and they can be set equal to zero.

The minimum $H^2$ of the quadratic form $F$ is equal in the given case to

$$H^2 = \frac{\Gamma(Y_1, \ldots, Y_p, R)}{\Gamma(Y_1, \ldots, Y_p)} \tag{2.8}$$

where the numerator and denominator are the Grammian determinants of the corresponding systems of vectors. Finally, we have the important relation

$$S = \sum_{i=1}^{p} c_i Y_i = -\frac{1}{\Gamma(Y_1, \ldots, Y_p)} \begin{vmatrix} (Y_1, Y_1) & \ldots & (Y_1, Y_p) & Y_1 \\ \cdot & \cdot & \cdot & \cdot \\ (Y_p, Y_1) & \ldots & (Y_p, Y_p) & Y_p \\ (R, Y_1) & \ldots & (R, Y_p) & 0 \end{vmatrix} \tag{2.9}$$

From Formulas (2.8) it follows that $H^2 = 0$ if $p = n$ and, hence $m \geqslant n$. In this case the number of the linearly independent vectors $Y_i$ is a maximum, and the vectors generate the entire $n$-dimensional space in which the vector $R$ lies.

A system of functions $u_i(t)$ is said to be essentially linearly independent on some set if the set of zeros of the function

$$c_1 u_1(t) + \ldots + c_m u_m(t) \qquad \text{when } c_1^2 + \ldots + c_m^2 \neq 0$$

is nowhere dense in this set.

We prove the next lemma for later use.

*Lemma.* If the system of functions $u_i(t)$ $(i = 1, \ldots, m;\ m \geqslant n)$ is essentially linearly independent on the interval $[0, T]$, then the set of points $t_0$ for which the rank of the matrix

$$\left\| \begin{matrix} y_1 & y_1' & \dots & y_1^{(n-1)} \\ y_2 & y_2' & \dots & y_2^{(n-1)} \\ \dots & \dots & \dots & \dots \\ y_m & y_m' & \dots & y_m^{(n-1)} \end{matrix} \right\| \tag{2.10}$$

is less than $n$ is a closed nowhere-dense set.

Indeed, let us assume that on the interval $t_1 \leqslant t \leqslant t_2$ the rank of the matrix (2.10) is less than $n$. We construct the differential equation

$$\left| \begin{matrix} y_1 & y_1' & \dots & y_1^{(n-1)} \\ \dots & \dots & \dots & \dots \\ y_{n-1} & y_{n-1}' & \dots & y_{n-1}^{(n-1)} \\ y & y' & \dots & y^{(n-1)} \end{matrix} \right| = \sum_{k=0}^{n-1} b_k(t)\, y^{(k)} = 0 \tag{2.11}$$

It is obvious that on the indicated interval all the $m$-functions $y_i(t)$ satisfy this $(n-1)$-order differential equation with continuous coefficients $b_k(t)$. But since $m \geqslant n$, the function $y_i(t)$ must be linearly dependent. This means that there exist constants $a_i$ such that

$$\alpha_1^2 + \ldots + \alpha_m^2 \neq 0, \qquad \alpha_1 y_1(t) + \ldots + \alpha_m y_m(t) \equiv 0$$

on the given interval. Since $L(y_i(t)) = u_i(t)$ ($L$ is the operator defined by Equation (1.1)), we have $a_1 u_1(t) + \ldots + a_m u_m(t) = 0$ everywhere on $[t_1, t_2]$.

The closure of the set under consideration follows from the fact that the complement of the set of points where the rank of the matrix (2.10) is equal to $n$ is, obviously, an open set.

We shall call the points $t_0$ where the rank of the matrix (2.10) is less than $n$ critical points; at these points one cannot eliminate the error $H$ by increasing the number of functions $u_i(t)$ to $n$ (or higher) even in the absence of any restrictions on the parameters $c_i$. We note that in concrete examples the critical points are distributed sparsely. From simple considerations it follows, for example, that if the functions $u_i(t)$ and the coefficients of Equation (1.1) are analytic, then the critical points are isolated. At the noncritical points, with $m = n$, we obtain the exact solution of the first problem by selecting the parameters $c_i$ in accordance with (2.7).

3. Let us now consider the first problem when the parameters $c_i$ are subjected to restrictions. We shall assume that the parameters $c_i$ are connected by the inequality

$$\rho(c_1, \ldots, c_m) \leqslant M \tag{3.1}$$

The points of the hyperplane $Q$ with radius vectors of the form $S = c_1 Y_1 + \ldots + c_m Y_m$, where the $c_i$ are connected by condition (3.1), fill some region $G$. Two cases can arise. In the first case, the minimum of the quadratic form $F$ is attained by the vector $S$ with its end $B$ inside the region $G$. In this case the system (2.7) yields the complete solution of the problem.

In the second case, the minimum form $F$ is attained on the boundary (Fig. 1) of the region $G$; it is obviously equal to the square of the distance from the point $A$ to the point $C$ on the boundary of the region $G$ and nearest to $A$. Since $AC^2 = CB^2 + AB^2$, and since the component $AB$ does not depend on the size and shape of the region $G$, the point $C$ is also the nearest point of the region $G$ from the point $B$. Thus, the error $AC$ with which the problem is solved has, seemingly, two components $CB$ and $AB$. Selecting a noncritical value $t_0$ and taking $m = n$, one can eliminate the component $AB$. Hence, one needs only to find ways for decreasing the length of the component $CB$.

This last task represents the problem on a conditional extremum; but when $m \geqslant n$ for the noncritical value $t_0$, one cannot deal directly with the extremum of the form $F$, for in this case the system of equations is a consistent set of simultaneous equations. Here, one may make use of Krein's [11] method.

Suppose $m \geqslant n$, and $t_0$ is a noncritical value. We shall consider the vector space $R_n$ generated by the $m$-dimensional vectors $Y^k(y_1^{(k)}, \ldots, y_m^{(k)})$ $(k = 0, 1, \ldots, n-1)$. Since $t_0$ is noncritical, the vectors $Y^k$ are linearly independent, and the dimensionality of the space $R_n$ is $n$. Alongside the space $R_n$ we consider the space $E_m$ of the vectors $C(c_1, \ldots, c_m)$. Suppose that in this space there is given a norm $\rho(C) = \rho(c_1, \ldots, c_m)$, i.e. a function satisfying the conditions

$$\rho(C) > 0, \quad \text{if} \quad C \neq 0; \quad \rho(\alpha C) = \alpha\rho(C), \quad \rho(C_1 + C_2) \leqslant \rho(C_1) + \rho(C_2) \tag{3.2}$$

The space $E_m$ can be considered [12, p. 113] as a space of linear functionals $\phi$ acting in $R$ according to the rule

$$\varphi(X) = \sum_{i=1}^{m} c_i x_i = (C, X) \qquad (X \subset R_n)$$

Hence, in $R_n$ as well as in the space $R_m$ which is an adjoint to the $E_m$ space, the norm $\|X\|$ is defined by the rule

$$\|X\| = \max(C, X) \text{ under the condition } \rho(C) = 1$$

The system (2.5) can now be represented in the form

$$(C, Y^k) = r^{(k)} \qquad (k = 0, 1, \ldots, n-1) \tag{3.3}$$

Let us determine the function $\lambda(t_0, m)$ by means of the conditions

$$\frac{1}{\lambda(t_0, m)} = \min\left\|\sum_{k=0}^{n-1} \gamma_k Y^k\right\| \quad \text{when } \sum_{k=0}^{n-1} \gamma_k r^{(k)} = 1 \tag{3.4}$$

From the basic result of [11] it follows that the system (3.3) has a solution $C(c_1, \ldots, c_m)$ satisfying the condition $\rho(c_1, \ldots, c_m) \leqslant M$ if, and only if, $\lambda(t_0, m) \leqslant M$. From the same work [11, p. 177] it also follows that the vector $\gamma_0 Y^0 + \gamma_1 Y^1 + \ldots + \gamma_{n-1} Y^{n-1}$ will be a minimizing vector of the problem (3.4) if, and only if, the vector $C(c_1, \ldots, c_m)$ satisfies the system (3.3) and also the conditions

$$\left|\sum_{i=1}^{m} c_i \sum_{k=0}^{n-1} \gamma_k y_i^{(k)}\right| = \lambda(t_0, m)\left\|\sum_{k=0}^{n-1} \gamma_k Y^k\right\|, \qquad \lambda(t_0, m) = \rho(c_1, \ldots, c_m) \tag{3.5}$$

From these results it follows first of all that the function $\lambda(t_0, m)$ is a continuous function $t_0$ on the set of noncritical points.

However, in contrast with the work of Kirillova [13], one may not assert that $\lambda(t_0, m)$ is a monotone function.

Let us show that the function $\lambda(t_0, m)$ is a nonincreasing function $m$. Indeed, suppose the minimizing elements of the problem (3.4) are

$$\gamma_0^\circ Y^0(m_0) + \gamma_1^\circ Y^1(m_0) + \ldots + \gamma_{n-1}^\circ Y^{n-1}(m_0) \text{ when } m = m_0$$

and $$\gamma_0' Y^0(m_1) + \gamma_1' Y^1(m_1) + \ldots + \gamma_{n-1}' Y^{n-1}(m_1) \text{ when } m = m_1 > m_0 \tag{3.6}$$

Obviously, we have

$$\|\gamma_0' Y^0(m_0) + \gamma_1' Y^1(m_0) + \ldots + \gamma_{n-1}' Y^{n-1}(m_0)\| \geqslant$$
$$\geqslant \|\gamma_0^\circ Y^0(m_0) + \gamma_1^\circ Y^1(m_0) + \ldots + \gamma_{n-1}^\circ Y^{n-1}(m_0)\| \tag{3.7}$$

But since the vector $\gamma_0' Y^0(m_0) + \gamma_1' Y'(m_0) + \ldots + \gamma_{n-1} Y^{n-1}(m_0)$ is the projection of the vector $\gamma_0' Y^0(m_1) + \gamma_1' Y'(m_1) + \ldots + \gamma_{n-1} Y^{n-1}(m_1)$ we have

$$\|\gamma_0' Y^0(m_1) + \gamma_1' Y^1(m_1) + \ldots + \gamma_{n-1}' Y^{n-1}(m_1)\| \geqslant$$
$$\geqslant \|\gamma_0^\circ Y^0(m_0) + \gamma_1^\circ Y^1(m_0) + \ldots + \gamma_{n-1}^\circ Y^{n-1}(m_0)\| \tag{3.8}$$

From relations (3.4) and the inequalities (3.7) and (3.8) we deduce that

$$\lambda(t_0, m_0) \geqslant \lambda(t_0, m_1)$$

which proves our assertion.

If the number $t_0$ is a critical number, then the vectors $Y^k$, $k = 0, 1, \ldots, (n = 1)$ will be linearly dependent. In this case it follows from (3.4) that $\lambda(t_0, m) = \infty$. Hence, the function $\lambda(t, m)$ will be a continuous function on the set of points where $\lambda(t, m) \leqslant M$. It follows from this that the equation $\lambda(t, m) = M$ has a smallest root $t_0$ which is not a critical number. This number $t_0$ gives us the optimum noncritical time of the transient process. We call attention to the fact that $t_0 \neq 0$, for the number 0 is always a critical value. With the equation $\lambda(t, m) = M$ one can also determine the minimum number $m$ of the control parameters $c_i$ for which there exists a solution of our problem.

**4.** Let us consider some examples of concrete metrics in the space $E_m$. We consider first the Euclidean metric, i.e. we set

$$\rho(c_1, \ldots, c_m) = \sqrt{c_1^2 + \ldots + c_m^2}$$

In the adjoint space $R_m$ the norm of the vector $X(x_1, \ldots, x_m)$ is defined by an analogous formula $\|X\| = \sqrt{(x_1^2 + \ldots + x_m^2)}$.

By means of (3.4) we find first $\lambda(t_0, m)$ from the relation

$$\frac{1}{\lambda(t_0, m)} = \min\left[\sum_{i=1}^{m}\left(\sum_{k=0}^{n-1}\gamma_k y_i^{(k)}(t_0)\right)^2\right]^{1/2} \quad \text{when} \sum_{k=0}^{n-1}\gamma_k r^{(k)}(t_0) = 1 \qquad (4.1)$$

After that, one must find the smallest root $t_0$ of the equation $\lambda(t_0, m) = M$, and then the corresponding values $\gamma_k$, $k = 0, 1, \ldots, (n-1)$ which yield the minimum (4.1). Since in the given case

$$\lambda(t_0, m) = \sqrt{c_1^2 + \ldots + c_m^2}$$

it follows from (3.4) that

$$\left|\sum_{i=1}^{m} c_i \sum_{k=0}^{n-1}\gamma_k y_i^{(k)}\right| = \left(\sum_{i=1}^{m} c_i^2\right)\left[\sum_{i=1}^{m}\left(\sum_{k=0}^{n-1}\gamma_k y_i^{(k)}\right)^2\right]^{1/2}$$

But the last equation will be valid when the numbers $c_i$ are proprotional to the numbers $\gamma_0 y_i^0 + \ldots + \gamma_{n-1} y_i^{(n-1)}$. Finally, we obtain

$$c_i = \lambda^2(t_0, m)\left(\gamma_0 y_i^{(0)} + \gamma_1 y_2^{(1)} + \ldots + \gamma_{n-1} y_i^{(n-1)}\right) \qquad (4.2)$$

Next, let us consider the case when the norm $\rho(c_1, \ldots, c_m)$ is defined by the formula

$$\rho(c_1, \ldots, c_m) = \max_{1 \leqslant i \leqslant m} |c_i|$$

In this case there is induced in $R_m$ a norm $X(x_1, \ldots, x_m)$ given by the formula

$$\|X\| = |x_1| + \ldots + |x_m|$$

We determine the function $\lambda(t_0, m)$ and the minimizing vector $\gamma_0 Y^0 + \gamma_1 Y^1 + \ldots + \gamma_{n-1} Y^{n-1}$ by means of the relations

$$\frac{1}{\lambda(t_0, m)} = \min \sum_{i=1}^{m} \left| \sum_{k=0}^{n-1} \gamma_k y_i^{(k)} \right|, \qquad \sum_{k=0}^{n-1} \gamma_k r^{(k)} = 1$$

From (3.4) it follows that the minimizing vector must satisfy the condition

$$\left| \sum_{i=1}^{m} c_i \sum_{k=0}^{n-1} \gamma_k y_i^{(k)} \right| = \max_{1 \leqslant i \leqslant m} |c_i| \sum_{i=1}^{m} \left| \sum_{k=0}^{n-1} \gamma_k y_i^{(k)} \right|$$

From this it follows that

$$c_i = \lambda(t_0, m) \operatorname{sign} \sum_{k=0}^{n-1} \gamma_k y_i^{(k)} \qquad (i = 1, \ldots, m)$$

Let us consider an example. Let the following equation be given:

$$\ddot{x} = c_1 \sin t + c_2 \cos t + c_3 \sin 2t + c_4 \cos 2t \tag{4.3}$$

It is required to transfer the point $x = 0$, $x' = 0$ on the straight line $x = 1$ into the point $x = 1$, $x' = 0$ under the restriction that $c_1^2 + c_2^2 + c_3^2 + c_4^2 \leqslant 1$.

This means that in the given problem $f(t) = 1$, $M = 1$. We obviously have

$$w_1(t, \tau) = 1, \qquad w_1'(t, \tau) = 0, \qquad w_0(t, \tau) = t - \tau, \qquad w_2'(t, \tau) = 1$$

We also have

$$y_1 = -\sin t, \qquad y_2 = -1 - \cos t, \qquad y_3 = -\frac{1}{4}\sin 2t, \qquad y_4 = \frac{1}{4}(1 - \cos 2t)$$

$$y_1' = -\cos t, \qquad y_2' = \sin t, \qquad y_3' = -\frac{1}{2}\cos 2t, \qquad y_4' = \frac{1}{2}\sin 2t$$

It is easily seen that the critical values of $t$ correspond to the points $t = 2k\pi$. The system (2.5) has the following form in this case:

$$-c_1\sin t+c_2(1-\cos t)-\frac{1}{4}c_3\sin 2t+\frac{1}{4}c_4(1-\cos 2t)=1$$
$$-c_1\cos t+c_2\sin t-\frac{1}{2}c_3\cos 2t+\frac{1}{2}c_4\sin 2t=0$$

Since $\rho(c_1, \ldots, c_4) = \sqrt{(c_1^2 + c_2^2 + c_3^2 + c_4^2)}$, $\lambda(t, 4) = \lambda(t)$ is given by the relation

$$\frac{1}{\lambda^2(t)}=\min\Big\{(\gamma_1\sin t+\gamma_2\cos t)^2+(\gamma_1(1-\cos t)+\gamma_2\sin t)^2+ + \left(\frac{1}{4}\gamma_1\sin 2t+\frac{1}{2}\gamma_2\cos 2t\right)^2+\left[\frac{1}{4}\gamma_1(1-\cos 2t)+\frac{1}{2}\gamma_2\sin 2t\right]^2\Big\}$$

under the condition that $\gamma_1 = 1$. It is not difficult to calculate that $\gamma_2 = -(8\sin t + \sin 2t)/10$ and

$$\frac{1}{\lambda^2(t)}=2-2\cos t+{}^1/_8(1-\cos 2t)-{}^1/_{80}(8\sin t+\sin 2t)^2=\Phi(t)$$

In order to have attainability it is necessary that $\lambda^2(t) \leqslant 1$, or $\Phi(t) \geqslant 1$. This condition is fulfilled, for example, when $t = \pi/2$. The shortest time for the [transient ] transfer process is found from the equation $\Phi(t) = 1$, which we shall not solve here. Knowing $\gamma_1$, $\gamma_2$ and $\lambda(t)$ we can easily find $c_i$ by Formula (4.2).

**5.** Let us proceed to the solution of the second problem. Taking into account the transformation $z = x - f(t)$, introduced in Section 2, and Equation (2.1), let us formulate the second problem in the following way: the problem is to find such parameters $c_i$ that the solution $z(t)$ of Equation (2.1) which satisfies the conditions $z^k(t_0) = 0$ $(k = 0, 1, \ldots, n-1)$ may approximate $z = 0$.

How the approximation is to be carried out in the $L_2$ space, i.e. on the basis of mean-square deviations, was shown in Section 1. Here, following the ideas of [5,6], we shall try to find ways of selecting the parameters $c_i$ which will diminish the maximum deviation of $z(t)$ from zero. From (2.1) we obtain, in analogy with (1.5)

$$z(t)=\int_{t_0}^{t}w_n(t,\tau)\left(\sum_{i=1}^{m}c_iu_i(t)-\varphi(\tau)\right)d\tau \qquad (L(f(\tau))=\varphi(\tau)) \tag{5.1}$$

Making use of the Buniakov-Schwarz inequality we obtain for $t_0 \leqslant t \leqslant T$

$$|z(t)|\leqslant N\left(\int_{t_0}^{T}\left(\sum_{i=1}^{m}c_iu_i(\tau)-\varphi(\tau)\right)^2d\tau\right)^{1/2} \tag{5.2}$$

Here

$$N=\max\left(\int_{t_0}^{t}w_n^2(t,\tau)\,d\tau\right)^{1/2} \qquad (t_0\leqslant t\leqslant T)$$

In the case when Equation (1.1) is an equation with constant coefficients we find that in accordance with [6]

$$N = \left(\int_{t_0}^{T} w_n^2(T, \tau)\, d\tau\right)^{1/2}$$

We select the parameters $c_i$ so that the integral

$$H^2 = \int_{t_0}^{T} \left(\sum_{i=1}^{m} c_i u_i(\tau) - \varphi(\tau)\right)^2 d\tau$$

will have, for a given $m$, the minimum value. In other words, we must find the best mean-square approximation of the function $\phi(t)$. We have already solved a similar problem in Section 1, and know that, for example, the $c_i$ must be found from the system of equations

$$\sum_{k=1}^{m} (u_i, u_k)\, c_k = (u_i, \varphi) \qquad (i = 1, \ldots, m) \tag{5.3}$$

If the system of functions $u_i(t)$, $i = 1, 2, \ldots,$ is a complete system, then by taking $m$ sufficiently large, one can make $H^2$ less than any given positive number.

We note now that the proposed method of approximation will be most effective over large intervals of time if the quantity $N$ in (5.2) is bounded as a function of $T$. This condition holds, for example, when $w_n(t, \tau)$ satisfies the condition

$$|w_n(t, \tau)| \leqslant Be^{-\alpha(t-\tau)} \qquad (\alpha > 0,\ B > 0)$$

which holds, in particular, in the case when the zero solution of the equation $L(z) = 0$ is stable according to the exponential law [14, p. 310].

Next, we shall consider the case when all the coefficients $c_i$ are bounded by the inequality

$$\rho(c_1, \ldots, c_m) \leqslant M \tag{5.4}$$

This gives rise to the problem of finding the best mean-square approximation of the function $\phi(t)$ under certain restrictions on the coefficients of the polynomial $c_1 u_1(t) + \ldots + c_m u_m(t)$. If the coefficients $c_i$ found by means of (5.3) do not satisfy condition (5.4), then the more natural method of finding the required set of coefficients consists of solving a problem of a conditional extremum. In this case, Lagrange's method leads to the system

$$\sum_{k=1}^{m}(u_i, u_k) = (u_i, \varphi) + \frac{\lambda}{2}\frac{\partial \rho}{\partial c_i} \qquad (i = 1, \ldots, m) \tag{5.5}$$

If the function $\rho(c_1, \ldots, c_m)$ satisfies conditions (3.2), then again, just as in Section 3, one can apply to this problem the method used in the $L$-problem of Krein.

Let us denote by $U_i$ the vector with the projections $(u_i, u_k)$, $k = 1, \ldots, m$. In accordance with [11], the system (5.3) has a solution $c_1, \ldots, c_m$ satisfying the inequality (5.4) if, and only if, the function $\lambda(m)$ defined by

$$\frac{1}{\lambda(m)} = \min\left\|\sum_{k=1}^{m}\gamma_k U_k\right\|, \qquad \sum_{k=1}^{m}\gamma_k(u_k, \varphi) = 1 \tag{5.6}$$

satisfies the inequality $\lambda(m) \leqslant M$.

Here the norm $\| U \|$ is defined in the space generated by the vectors $U_i$ as in a space which is adjoint to the space $E_m$ of vectors $C(c_1, \ldots, c_m)$ with the norm $\rho(c_1, \ldots, c_m)$.

Let us note now that if the functions $u_i(t)$ form an orthonormal system on the interval $[t_0, T]$, i.e. they satisfy the condition $(u_i, u_k) = 0$ when $i \neq k$ and $(u_i, u_i) = 1$, then the problem under consideration can be solved quite simply. Indeed, in this case the system (5.3) yields $c_k = (u_k, \phi_k)$ $(k = 1, \ldots, m)$.

For an arbitrary system of coefficients $b_k$ we have

$$H^2 = \int_{t_0}^{T}\left(\sum_{i=1}^{m} b_i u_i(t) - \varphi(t)\right)^2 dt = \sum_{i=1}^{m}(b_i - c_i)^2 + \int_{t_0}^{T}\varphi^2(t)\,dt - \sum_{i=1}^{m} c_i^2$$

Suppose that the $b_k$ satisfy the inequality (5.4). The difference

$$\int_{t_0}^{T}\varphi^2(t)\,dt - \sum_{i=1}^{m} c_i^2$$

does not depend on the shape of the region (5.4); it can be diminished only by increasing $m$. For a given $m$ one can decrease $H^2$ only by decreasing the quantity $h^2 = (b_1 - c_1)^2 + \ldots + (b_m - c_m)^2$, which in the space of the parameters is equal to the distance from the point $C(c_1, \ldots, c_m)$ to the point $B(b_1, \ldots, b_m)$ lying in the region (5.4).

But this means that $h^2$ will be a minimum when we choose for the point

$B(b_1, \ldots, b_m)$ the point in the region (5.4) that is nearest to $C$. (The author is indebted to S.B. Stechkin for this observation).

If, for example, $\rho(c_1, \ldots, c_m) = \sqrt{(c_1^2 + \ldots + c_m^2)}$, then it follows from the indicated considerations that

$$b_i = M \frac{c_i}{\sqrt{c_1^2 + \ldots + c_m^2}}, \qquad \text{if } c_1^2 + \ldots + c_m^2 > M^2$$

$$b_i = c_i, \qquad \text{if } c_1^2 + \ldots + c_m^2 \leqslant M^2$$

If, however, $\rho(c_1, \ldots, c_m) = \max |c_i|$ when $1 \leqslant i \leqslant m$, then

$$b_i = M \operatorname{sign} c_i \text{ when } \rho(c_1, \ldots, c_m) > M, \qquad b_i = c_i \text{ when } \rho(c_1, \ldots, c_m) \leqslant M$$

By developing further these considerations one can obtain a new method for solving problems in the general case of a non-orthonormalized system. It is known [8, p. 320] that an arbitrary system of linearly independent functions can be orthonormalized. Let us suppose that the process of orthonormalization yields the system of functions $\{v_i(t)\}$. Then the polynomial $c_1 u_1(t) + \ldots + c_m u_m(t)$ will be transformed into the polynomial $b_1 v_1(t) + \ldots + b_m v_m(t)$, whereby the coefficients $c_i$ become linear functions of the coefficients $b_i$. Condition (5.4) takes on the form of a restriction on the coefficients $b_i$. Hereby, the region $G$, given by the inequality (5.4), is transformed linearly into a new region $G_1$ in the space of the coefficients $b_i$. The problem is thus reduced, obviously, to finding within the region $G_1$ the point nearest to the point with the coordinates equal to the Fourier coefficients.

In conclusion, let us remark that the above considerations are applicable to any problem in which it is necessary to find the best mean-square approximation under restrictions on the coefficients of the polynomial. Thus, returning to the problem solved in Section 1, one can render it more complicated and look for a solution of Equation (1.1) in the form of a mean-square approximation of the function $f(t)$ under a restriction on the initially given parameters $c_i$.

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